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ON THE STATE OF STRESS NEAR CURVILINEAR HOLES IN SHELLS

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ON THE STATE OF STRESS NEAR CURVILINEAR HOLES IN SHELLS

G. N. Savin and A. N. Guz'

ABSTRACT

The investigation of stress concentrations near holes in shells is reduced to the solution of boundary value problems in which the variables are not separated either in the solving equations or in the boundary conditions (ref. 1). An exception to this is the spherical shell with a circular hole (ref. 2) and a cylindrical shell with a small circular hole (ref. 3). As a result of this situation it is not possible to obtain an accurate solution of the problem for other hole shapes.

The present work proposes an approximate method of investigating the state of stress in shells near curvilinear holes of arbitrary shape based on the method of "perturbed boundary form" (ref. 4). For certain specific hole shapes this method is considered in reference 5. This method was applied by S. G. Lekhnitskiy (ref. 6) in the case of a two-dimensional problem of elasticity involving an anistropic medium. By applying the proposed method we obtain a sequence of boundary value problems for a circular hole.

In the case of a spherical shell with large holes and in case of cylindrical holes weakened with small holes, we are led to problems in which the variables are separable when holes are of arbitrary shape but do not contain angular points.

The state of stress in a shell weakened by a hole is represented as a sum (ref. 1):[1] of the state of stress ($T_n{}^0, \ldots, Q_s{}^0$) in the continuous shell /96*

which has not been weakened by the hole using the same boundary conditions and an additional state of stress involving the perturbations ($T_n, \ldots, Q_s$) produced

*Numbers given in margin indicate pagination in original foreign text.

[1]This breakdown of the state of stress is possible because the problem is linear.

by the presence of the considered hole. The initial state of stress ($T_n^0, \ldots, Q_s^0$) is called the basic state and is assumed to be known.

The problem is to find the additional state of stress ($T_n, \ldots, Q_s$).

Experimental investigations (ref. 1) show that the perturbations near the hole with a contour devoid of angular points have a local nature and are attenuated very rapidly at distances away from the hole. To determine the components ($T_n, \ldots, Q_s$) we can apply the theory of the state of stress with a large variability index. In this case the solution of the formulated problem is reduced (ref. 1) to the integration of the differential equation

$$\nabla^2\nabla^2\Phi - i\varkappa^2\nabla_k^2\Phi = 0$$
$$\Phi = w + in\varphi, \qquad n = \frac{\sqrt{12(1-\nu^2)}}{Eh^2}, \qquad \varkappa = r_0\left(\frac{12(1-\nu^2)}{h^2}\right)^{1/4} \tag{1.1}$$

with the corresponding boundary conditions. Here r_0 is a real quantity which characterizes the dimensions of the hole. Equation (1.1) is represented in dimensionless coordinates with respect to r_0.

Let us consider the plane of variables associated with the surface of the shell. Figure 1 shows the following: x,y--the orthogonal system of coordinates; r,θ--polar system of coordinates; Γ--the contour of the hole in the plane of the variables associated with the surface of the shell; n--the unit vector of the normal to contour; ψ--the angle between the normal and radial direction (fig. 1). The solution of equation (1.1) must satisfy the corresponding boundary conditions on the contour Γ and must satisfy certain conditions at infinity (ref. 1). /97

2. Let us assume that the contour Γ has a shape such that the function

$$z = \omega(\zeta), \qquad \omega(\zeta) = \zeta + \varepsilon f(\zeta)$$
$$(z = re^{i\theta}, \ \zeta = \rho a^{i\nu}, \ \varepsilon \ll 1) \tag{2.1}$$

produces a conformal transformation of the infinite plane with a circular hole of unit radius into an infinite plane with a hole bounded by the contour Γ. The function $f(\zeta)$ depends on the form of Γ, $\epsilon \ll 1$, and the roots of the equation

$$1 + \varepsilon f'(\zeta) = 0$$

must lie in the plane ζ inside the circle of unit radius. With these limitations the function $\omega(\zeta)$ (2.1) produces a conformal transformation of the region

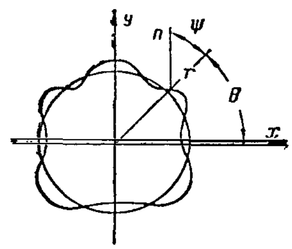

Figure 1

external to the unit circle on the region external to the contour Γ (fig. 1). In order that in each of the subsequent approximations we have a boundary problem with separable variables, it is necessary that the solution of equation (1.1) be representable in the form

$$\Phi(r,\ \theta) = \sum_{k=0}^{\infty} f_k(r)\cos k\theta + g_k(r)\sin k\theta \tag{2.2}$$

The values r, θ, ψ on the contour Γ are equal when ρ = 1

$$r = \sqrt{\zeta\bar{\zeta} + \varepsilon[\bar{\zeta}f(\zeta) + \zeta\overline{f(\zeta)}] + \varepsilon^2 f(\zeta)\overline{f(\zeta)}}$$
$$\theta = \operatorname{arc\,tg}\frac{\sin\gamma + (\varepsilon/\rho)\operatorname{Im} f(\zeta)}{\cos\gamma + (\varepsilon/\rho)\operatorname{Re} f(\zeta)}, \qquad e^{2i\psi} = \frac{\zeta^2[\overline{\omega(\zeta)}]^2[\omega'(\zeta)]^2}{\rho^2|\overline{\omega(\zeta)}|^2|\omega'(\zeta)|^2} \tag{2.3}$$

We represent the solution of equation (1.1) and the components of the state of stress and deformation on the contour Γ as a series of the parameter ε

$$\Phi(r,\ \theta) = \sum_{j=0}^{\infty}\varepsilon^j\Phi_j(r,\ \theta) \tag{2.4}$$

$$T_n|_\Gamma = \sum_{j=0}^{\infty}\varepsilon^j T_n^{(j)}, \qquad T_s|_\Gamma = \sum_{j=0}^{\infty}\varepsilon^j T_s^{(j)}, \qquad S_{ns}|_\Gamma = \sum_{j=0}^{\infty}\varepsilon^j S_{ns}^{(j)}$$
$$G_n|_\Gamma = \sum_{j=0}^{\infty}\varepsilon^j G_n^{(j)}, \qquad G_s|_\Gamma = \sum_{j=0}^{\infty}\varepsilon^j G_s^{(j)}, \qquad H_{ns}|_\Gamma = \sum_{j=0}^{\infty}\varepsilon^j H_{ns}^{(j)} \tag{2.5}$$
$$Q^\circ{}_n|_\Gamma = \sum_{j=0}^{\infty}\varepsilon^j Q^\circ{}_n^{(j)} \qquad w|_\Gamma = \sum_{j=0}^{\infty}\varepsilon^j w_j, \qquad \left.\frac{\partial w}{\partial n}\right|_\Gamma = \sum_{j=0}^{\infty}\varepsilon^j\left(\frac{\partial w}{\partial n}\right)^{(j)}$$
$$u_n|_\Gamma = \sum_{j=0}^{\infty}\varepsilon^j u_n^{(j)}, \qquad u_s|_\Gamma = \sum_{j=0}^{\infty}\varepsilon^j u_s^{(j)}$$

With the solution in form (2.2), we can use known equations to find /98
the stress and strain components in the polar system of coordinates. To determine the corresponding components in the coordinate system ρ, γ, obtained

from (2.1) when[1] $\rho = 1$, we use the corresponding transformation equations (ref. 1) for the stress and strain components when transforming from one system of coordinates to another (fig. 1). The values of the quantities r, θ and ψ from (2.3) should be substituted into the expressions which are obtained. Substituting (2.4) into (1.1) and referring initially to the polar system of coordinates, we obtain an infinite system of equations which has the following form the j-th approximation

$$\nabla^2\nabla^2\Phi_j(r,\theta) - i\varkappa^2\nabla_k^2\Phi_j(r,\theta) = 0 \qquad (j = 0, 1, 2, \ldots) \tag{2.6}$$

In accordance with (2.2), the solution of (2.6) may be written in the form

$$\Phi_j(r,\theta) = \sum_{k=0}^{\infty} f_{kj}(r)\cos k\theta + g_{kj}(r)\sin k\theta \tag{2.7}$$

Relationships (2.3) are also expanded in series of ϵ, while the stress and strain components at the contour Γ are determined by the method specified above. For the j-th approximation we obtain

$$\begin{aligned}
T_n^{(j)} &= T_r^{(j)}|_{\rho=1} + \sum_{m=0}^{j-1} [L_1^{(j-m)} T_r^{(m)} + L_2^{(j-m)} (T_\theta^{(m)} - T_r^{(m)}) + L_3^{(j-m)} S_{r\theta}^{(m)}]|_{\rho=1} \\
T_s^{(j)} &= T_\theta^{(j)}|_{\rho=1} + \sum_{m=0}^{j-1} [L_1^{(j-m)} T_\theta^{(m)} + L_2^{(j-m)} (T_r^{(m)} - T_\theta^{(m)}) - L_3^{(j-m)} S_{r\theta}^{(m)}]|_{\rho=1} \\
S_{ns}^{(j)} &= S_{r\theta}^{(j)}|_{\rho=1} + \sum_{m=0}^{j-1} \Big[(L_1^{(j-m)} - 2L_2^{(j-m)}) S_{r\theta}^{(m)} + \tfrac{1}{2} L_3^{(j-m)} (T_\theta^{(m)} - T_r^{(m)})\Big]_{\rho=1} \\
G_n^{(j)} &= G_r^{(j)}|_{\rho=1} + \sum_{m=0}^{j-1} [L_1^{(j-m)} G_r^{(m)} + L_2^{(j-m)} (G_\theta^{(m)} - G_r^{(m)}) + L_3^{(j-m)} H_{r\theta}^{(m)}]|_{\rho=1} \\
G_s^{(j)} &= G_\theta^{(j)}|_{\rho=1} + \sum_{m=0}^{j-1} [L_1^{(j-m)} G_\theta^{(m)} + L_2^{(j-m)} (G_r^{(m)} - G_\theta^{(m)}) - L_3^{(j-m)} H_{r\theta}^{(m)}]|_{\rho=1} \\
H_{ns}^{(j)} &= H_{r\theta}^{(j)}|_{\rho=1} + \\
&\quad + \sum_{m=0}^{j-1} \Big[(L_1^{(j-m)} - 2L_2^{(j-m)}) H_{r\theta}^{(m)} + \frac{1}{2} L_3^{(j-m)} (G_\theta^{(m)} - G_r^{(m)}]\Big)|_{\rho=1} \\
Q_n^{\bullet} &= -\frac{D}{r_0^3}\Big[\frac{\partial}{\partial\rho}\nabla^2 + \frac{1-\nu}{\rho}\frac{\partial^3}{\partial\rho\,\partial\gamma^2}\frac{1}{\rho}\Big]\operatorname{Re}\Phi_j(\rho,\gamma)|_{\rho=1} - \\
&\quad - \frac{D}{r_0^3}\sum_{m=0}^{j-1} L_4^{(j-m)} \operatorname{Re}\Phi_m(\rho,\gamma)|_{\rho=1} \\
w_j &= \operatorname{Re}\Phi_j(\rho,\gamma)|_{\rho=1} + \sum_{m=0}^{j-1} L_1^{(j-m)} \operatorname{Re}\Phi_m(\rho,\gamma)|_{\rho=1}
\end{aligned} \tag{2.8}$$

[1]The coordinate line $\rho = 1$ in view of (2.1) is the contour of the hole.

$$\left(\frac{\partial w}{\partial n}\right)^{(j)} = \frac{1}{r_0}\frac{\partial}{\partial \rho}\operatorname{Re}\Phi_j(\rho,\ \gamma)\,|_{\rho=1} +$$

$$+\frac{1}{r_0}\sum_{m=0}^{j-1}\left[L_5^{(j-m)}\frac{\partial}{\partial\rho} + L_6^{(j-m)}\frac{1}{\rho}\frac{\partial}{\partial\gamma}\right]\operatorname{Re}\Phi_m(\rho,\ \gamma)\,|_{\rho=1}$$

$$u_n^{(j)} = u^{(j)}\,|_{\rho=1} + \sum_{m=0}^{j-1}[L_5^{(j-m)}u^{(m)} + L_6^{(j-m)}v^{(m)}]\,|_{\rho=1}$$

$$u_s^{(j)} = v^{(j)}\,|_{\rho=1} + \sum_{m=0}^{j-1}[L_5^{(j-m)}v^{(m)} - L_6^{(j-m)}u^{(m)}]\,|_{\rho=1}$$

All of the quantities with indices j and m which are in the right sides of (2.8) have the following form /99

$$T_r^{(m)} = \frac{1}{nr_0^2}\left(\frac{1}{\rho^2}\frac{\partial^2}{\partial\gamma^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\right)\operatorname{Im}\Phi_m(\rho,\ \gamma), \qquad T_\theta^{(m)} = \frac{1}{nr_0^2}\frac{\partial^2}{\partial\rho^2}\operatorname{Im}\Phi_m(\rho,\ \gamma)$$

$$S_{r\theta}^{(m)} = -\frac{1}{nr_0^2}\frac{\partial^2}{\partial\rho\,\partial\gamma}\frac{\operatorname{Im}\Phi_m(\rho,\ \gamma)}{\rho}$$

$$G_r^{(m)} = -\frac{D}{r_0^2}\left[(1-\nu)\frac{\partial^2}{\partial\rho^2} + \nu\nabla^2\right]\operatorname{Re}\Phi_m(\rho,\ \gamma) \tag{2.9}$$

$$G_\theta^{(m)} = -\frac{D}{r_0^2}\left[\nabla^2 - (1-\nu)\frac{\partial^2}{\partial\rho^2}\right]\operatorname{Re}\Phi_m(\rho,\ \gamma)$$

$$H_{r\theta}^{(m)} \doteq -D\frac{1-\nu}{r_0^2}\frac{\partial^2}{\partial\rho\,\partial\gamma}\frac{\operatorname{Re}\Phi_m(\rho,\ \gamma)}{\rho}$$

The displacements $u^{(m)}$ and $v^{(m)}$ are obtained by integrating the systems of equations

$$\frac{\partial u^{(m)}}{\partial\rho} = -r_0\frac{\operatorname{Re}\Phi_m(\rho,\ \gamma)}{R_r'} + \frac{1}{Ehnr_0}\left[\nabla^2 - (1+\nu)\frac{\partial^2}{\partial\rho^2}\right]\operatorname{Im}\Phi_m(\rho,\ \gamma)$$

$$\frac{1}{\rho}\frac{\partial v^{(m)}}{\partial\gamma} + \frac{u^{(m)}}{\rho} = -r_0\frac{\operatorname{Re}\Phi_m(\rho,\ \gamma)}{R_\theta'} + \frac{1}{Ehnr_0}\left[(1+\nu)\frac{\partial^2}{\partial\rho^2} - \nu\nabla^2\right]\operatorname{Im}\Phi_m(\rho,\ \gamma) \tag{2.10}$$

$$\frac{1}{\rho}\frac{\partial u^{(m)}}{\partial\gamma} + \rho\frac{\partial}{\partial\rho}\left(\frac{v^{(m)}}{\rho}\right) = 2r_0\frac{\operatorname{Re}\Phi_m(\rho,\ \gamma)}{R_{12}} - 2\frac{1+\nu}{Ehnr_0}\frac{\partial^2}{\partial\rho\,\partial\gamma}\frac{\operatorname{Im}\Phi_m(\rho,\ \gamma)}{\rho}$$

Equations (2.8) also contain six differential operators $L_k^{(j-m)}$ $(k = 1, 2, \ldots, 6)$, whose order is indicated by the superscript. We write in expanded form the operators $L_k^{(j-m)}$, which will be necessary in solving the formulated problem for the zero, first and second approximation

$$L_1^{(1)} = \frac{\zeta\overline{f(\zeta)} + \bar{\zeta}f(\zeta)}{2\rho}\frac{\partial}{\partial\rho} + \left(\frac{f(\zeta) - \overline{f(\zeta)}}{2i\rho}\cos\gamma - \frac{f(\zeta) + \overline{f(\zeta)}}{2\rho}\sin\gamma\right)\frac{\partial}{\partial\gamma}, \qquad L_2^{(1)} = 0$$

$$L_1^{(2)} = \frac{[\zeta\overline{f(\zeta)} + \bar{\zeta}f(\zeta)]^2}{8\rho^2}\frac{\partial^2}{\partial\rho^2} + \frac{\zeta\overline{f(\zeta)} + \bar{\zeta}f(\zeta)}{\rho^2}\times$$

$$\times\left(\frac{f(\zeta)-\overline{f(\zeta)}}{4i}\cos\gamma-\frac{f(\zeta)+\overline{f(\zeta)}}{4}\sin\gamma\right)\frac{\partial^2}{\partial\rho\,\partial\gamma}+$$

$$+\frac{i\,[f^2(\zeta)+\overline{f^2(\zeta)}]\sin 2\gamma-[f^2(\zeta)-\overline{f^2(\zeta)}]\cos 2\gamma}{4i\rho^2}\frac{\partial}{\partial\gamma}-\frac{[\zeta\overline{f(\zeta)}-\bar\zeta f(\zeta)]^2}{8\rho^3}\frac{\partial}{\partial\rho}+$$

$$+\frac{2if(\zeta)\overline{f(\zeta)}-i\,[f^2(\zeta)+\overline{f^2(\zeta)}]\cos 2\gamma-[f^2(\zeta)-\overline{f^2(\zeta)}]\sin 2\gamma}{8i\rho^2}\frac{\partial^2}{\partial\gamma^2}$$

$$L_2^{(2)}=-\frac{1}{4}[f'(\zeta)-\overline{f'(\zeta)}]^2-\frac{[\zeta\overline{f(\zeta)}-\bar\zeta f(\zeta)]^2+2\zeta\bar\zeta\,[\zeta\overline{f(\zeta)}-\bar\zeta f(\zeta)]\,[f'(\zeta)-\overline{f'(\zeta)}]}{4\zeta^2\bar\zeta^2}$$

$$L_3^{(1)}=\frac{1}{i}[f'(\zeta)-\overline{f'(\zeta)}]+\frac{\zeta\overline{f(\zeta)}-\bar\zeta f(\zeta)}{i\zeta\bar\zeta}$$

$$L_3^{(2)}=\frac{\bar\zeta^2f^2(\zeta)-\zeta^2\overline{f^2(\zeta)}+\zeta^2\bar\zeta^2\,[\overline{f'^2(\zeta)}-f'^2(\zeta)]}{2i\zeta^2\bar\zeta^2}+\left[\frac{f'(\zeta)-\overline{f'(\zeta)}}{i}+\frac{\zeta\overline{f(\zeta)}-\bar\zeta f(\zeta)}{i\zeta\bar\zeta}\right]\times$$

$$\times\left[\frac{\bar\zeta f(\zeta)+\zeta\overline{f(\zeta)}}{2\rho}\frac{\partial}{\partial\rho}+\left(\frac{f(\zeta)-\overline{f(\zeta)}}{2i\rho}\cos\gamma-\frac{f(\zeta)+\overline{f(\zeta)}}{2\rho}\sin\gamma\right)\frac{\partial}{\partial\gamma}\right] \tag{2.11}$$

$$L_4^{(1)}=\left[L_1^{(1)}\frac{\partial}{\partial\rho}+\frac{1}{2}L_3^{(1)}\frac{1}{\rho}\frac{\partial}{\partial\gamma}\right]\nabla^2+\frac{1-\nu}{\rho}\left[\left(\frac{\partial}{\partial\gamma}L_1^{(1)}-\frac{f'(\zeta)+\overline{f'(\zeta)}}{2}\frac{\partial}{\partial\gamma}\right)\frac{\partial^2}{\partial\rho\,\partial\gamma}\frac{1}{\rho}+\right.$$

$$\left.+\frac{1}{2}\frac{\partial}{\partial\gamma}L_3^{(1)}\left(\nabla^2-2\frac{\partial^2}{\partial\rho^2}\right)\right]$$

$$L_4^{(2)}=\left[\left(L_1^{(2)}-\frac{1}{2}L_2^{(2)}\right)\frac{\partial}{\partial\rho}+\frac{1}{2}L_3^{(2)}\frac{1}{\rho}\frac{\partial}{\partial\gamma}\right]\nabla^2+$$

$$+\frac{1-\nu}{\rho}\left\{\left[\frac{3f'^2(\zeta)+2f'(\zeta)\overline{f'(\zeta)}+3\overline{f'^2(\zeta)}}{8}\frac{\partial}{\partial\gamma}-\frac{f'(\zeta)+\overline{f'(\zeta)}}{2}\frac{\partial}{\partial\gamma}L_1^{(1)}+\right.\right.$$

$$\left.\left.+\frac{\partial}{\partial\gamma}(L_1^{(2)}-2L_2^{(2)})\right]\frac{\partial^2}{\partial\rho\,\partial\gamma}\frac{1}{\rho}+\frac{1}{2}\left[\frac{\partial}{\partial\gamma}L_3^{(2)}-\frac{f'(\zeta)+\overline{f'(\zeta)}}{2}\frac{\partial}{\partial\gamma}L_3^{(1)}\right]\nabla^2-2\frac{\partial^2}{\partial\rho^2}\right)\right\}$$

$$L_5^{(1)}=L_1^{(1)},\qquad L_6^{(1)}=\frac{1}{2}L_3^{(1)},\qquad L_5^{(2)}=L_1^{(2)}-\frac{1}{2}L_2^{(2)},\qquad L_6^{(2)}=\frac{1}{2}L_3^{(2)}$$

By using relationships (2.3) we expand the right sides of the boundary/100 conditions on the contour Γ in series of ϵ. The function $\Phi_m(\rho,\gamma)$ in (2.8)-(2.10) should be interpreted as the solution of equation (2.6) in the m-th approximation, in which r is replaced by ρ and θ is replaced by γ. In the j-th approximation only the function $\Phi_j(\rho,\gamma)$ is unknown while the functions $\Phi_m(\rho,\gamma)$ $(m < j)$ are functions known from the preceding approximation. Introducing the expansions of the corresponding components from (2.8) into the considered boundary conditions, we obtain a system of equations for determining the coefficients in functions $f_{kj}(r)$ and $g_{kj}(r)$ from (2.7).

We can see from (2.8) and (2.9) that in each of the successive approximations (for each j) the problems are formally reduced to problems for a circular hole in the plane ζ. Thus the solution of the problem for the hole of arbitrary shape is reduced to a solution of a series of boundary value problems for a circular hole.

The expanded form $f(\zeta)$ in (2.1) for different holes is taken from the function $\omega(\zeta)$ (refs. 6 and 7) which gives a conformal transformation of the region external to the unit circle on the region external to the hole of considered shape.

The value of any quantity, for example $T_s^*|\Gamma$, obtained in the n-th approximation will represent the following quantity

$$T^*_{s(n)}|_\Gamma = T_s^0|_\Gamma + \sum_{j=0}^{n} \varepsilon^j T_\theta^{(j)}\Big|_{\rho=1} + \sum_{j=0}^{n}\sum_{m=0}^{j-1} \varepsilon^j [L_1^{(j-m)} T_\theta^{(m)} + \\ + L_2^{(j-m)} (T_r^{(m)} - T_\theta^{(m)}) - L_3^{(j-m)} S_{r\theta}^{(m)}]\Big|_{\rho=1} \tag{2.12}$$

For a spherical shell $R_r' = R_\theta' = R$, equation (2.6) will be /101

$$\nabla^2\nabla^2\Phi_j(r, \theta) - i\varkappa^2\nabla^2\Phi_j(r, \theta) = 0 \qquad (\varkappa = r_0\sqrt[4]{12(1-\nu^2)/R^2h^2}) \tag{3.1}$$

The solution of equation (3.1) which satisfies the conditions at infinity (ref. 1) will have the form

$$\Phi_j(r, \theta) = iB_{j0}\ln r + (C_{j0} + iD_{j0}) H_0^{(1)}(r\varkappa\sqrt{-i}) + \sum_{k=1}^{\infty} [(A_{jk} + iB_{jk}) r^{-k} + \\ + (C_{jk} + iD_{jk}) H_k^{(1)}(r\varkappa\sqrt{-i})] \frac{\cos k\theta}{\sin k\theta} \\ (H_k^{(1)}(r\varkappa\sqrt{-i}) - her_k\varkappa r + ihei_k\varkappa r) \tag{3.2}$$

Here A_{jk}, B_{jk}, C_{jk} and D_{jk} are constants of the Hankel function of the first kind and of the k-th order (ref. 8). Solution in the form (3.2) makes it possible to obtain a solution for a problem involving a spherical shell weakened by holes of arbitrary shapes whose contours do not have angular points.

As an illustration we shall present the value of the stress concentration coefficient k for a hole of near elliptical size for various values a/b in a spherical shell loaded with a uniform external pressure (ref. 5). We assume that the basic state of stress is momentless; it is assumed that the hole is closed with a cover which transmits only the transverse force. Assuming that for the shell R = 250 cm, h = 0.3 cm, $r_0 = 1/2(a+b) = 10.5$ cm, $\nu = 0.3$; we obtain the following value for the coefficient k

$$k = \frac{T_s}{p_0 h}\Big|_\Gamma = 4.29 + \frac{a-b}{a+b} 12.77\cos 2\gamma, \qquad \varepsilon = \frac{a-b}{a+b} \tag{3.3}$$

The values represented in table 1 have turned out to be equal.

$a/b = 1.0$	1.1	1.2	1.3	1.4	1.5
$k\|_{\gamma=0} = 4.29$	4.91	5.46	5.95	6.46	6.85
$k\|_{\gamma=1/2\pi} = 4.29$	3.63	3.13	2.63	2.12	1.74

4. Let us consider the case of a circular cylindrical shell.

If in place of φ we introduce φ and Φ unchanged $\Phi = w + i n\varphi$, equation (2.6) for the cylindrical shell will take on the form

$$\nabla^2\nabla^2\Phi_j(r, \theta) + 8i\beta^2 \frac{\partial^2}{\partial x^2}\Phi_j(r, 0) = 0, \qquad \beta = \frac{r_0}{\sqrt{Rh}}\frac{(3(1-\nu^2))^{1/4}}{2} \tag{4.1}$$

In expressions (2.9) and (2.10) the sign in front of $\mathrm{Im}\Phi_m(\rho, \gamma)$ should be reversed. Here the ox-axis coincides with the direction of the generatrix, while the oy-axis coincides with the directrix. This change in designations is made so that they coincide with those assumed in references 3 and 9. Equation (4.1) cannot be represented in the form (2.7) (ref. 3), however, for small holes (with an accuracy up to β^2) the unknown solution may be represented in the form

$$\Phi_j(r, \theta) = \sum_{k=0}^{\infty} f_{kj}(r)\begin{matrix}\cos k\theta\\ \sin k\theta\end{matrix} + \beta^2\sum_{k=0}^{\infty} g_{kj}(r)\begin{matrix}\cos k\theta\\ \sin k\theta\end{matrix} \tag{4.2}$$

Using the results of reference 9, we write the solution of (4.1) in the form (4.2) for a biaxial state of stress when the hole has two axes of symmetry coinciding in direction with the generatrix and the directrix /102

$$\begin{aligned}\mathrm{Im}\,\Phi_j(r,\theta) &= \frac{2}{\pi}A_0^{(j)}(\ln r + \gamma') + \frac{\cos 2\theta}{\pi}\left(A_2^{(j)} + \frac{B_2^{(j)}}{r^2}\right) + \\ &+ \sum_{k=4,6,\ldots}^{\infty}\left(\frac{k-2}{2r^{k-2}}A_k^{(j)} + \frac{B_k^{(j)}}{r^k}\right)\frac{\cos k\theta}{\pi} + \frac{1}{\pi}\beta^2\Big\{2B_0^{(j)}(\ln r + \gamma') - \\ &- \frac{\pi}{4}(2A_0^{(j)} + A_2^{(j)})r^2 + \left[-\frac{\pi}{4}(A_0^{(j)} + A_2^{(j)})r^2 + E_2^{(j)} + \frac{F_2^{(j)}}{r^2}\right]\cos 2\theta + \\ &+ \sum_{k=4,6,\ldots}^{\infty}\left(\frac{E_k^{(j)}}{r^{k-2}} + \frac{F_k^{(j)}}{r^k}\right)\cos k\theta\Big\}, \quad \gamma' = \frac{\ln\gamma\beta}{\sqrt{2}} \quad (\gamma \text{ - Euler's constant})\end{aligned} \tag{4.3}$$

$$\begin{aligned}\mathrm{Re}\,\Phi_j(r,\theta) &= \frac{1}{\pi}\beta^2\Big\{-2D_0^{(j)}(\ln r + \gamma') + (2A_0^{(j)} + A_2^{(j)})r^2(\ln r + \gamma') - \\ &- \frac{1}{4}(4A_0^{(j)} + A_2^{(j)})r^2 + \Big[(A_0^{(j)} + A_2^{(j)})r^2(\ln r + \gamma') - \frac{A_2^{(j)}}{6}r^2 + H_2^{(j)} +\end{aligned}$$

$$+\frac{1}{r^2}K_2^{(j)}\Big]\cos 2\theta+\frac{1}{12}A_2^{(j)}r^2\cos 4\theta+$$

$$+\sum_{k=4,6,\ldots}^{\infty}\left(\frac{k-4}{24r^{k-6}}A_{k-2}^{(j)}+\frac{B_{k-2}^{(j)}-A_k^{(j)}}{4r^{k-4}}+\frac{H_k^{(j)}}{r^{k-2}}+\frac{K_k^{(j)}}{r^k}\right)\cos k\theta\Big\}$$

Assuming that the basic state of stress is momentless and equal to $T_x^{\circ} = ph$, $T_y^{\circ} = qh$, $S_{xy}^{\circ} = \tau h$, we write the components of the basic state of stress in the cylindrical shell, for accuracy up to the term ϵ^2, in the form

$$\begin{aligned}
T_n^{\circ} &= {}^1/_2(p+q)h+{}^1/_2(p-q)h\cos 2\gamma+\tau h\sin 2\gamma+{}^1/_2\,\varepsilon\,[f'(\zeta)-\overline{f'(\zeta)}]\times\\
&\times[{}^1/_2(p-q)h(\zeta^2-\bar{\zeta}^2)-i\tau h(\zeta^2+\bar{\zeta}^2)]-{}^1/_2\,\varepsilon^2[f'(\zeta)-\overline{f'(\zeta)}]\times\\
&-\{{}^1/_2(p-q)h(\zeta^2\overline{f'(\zeta)}-\bar{\zeta}^2f'(\zeta))-i[\zeta^2\overline{f'(\zeta)}+\bar{\zeta}^2f'(\zeta)]\tau h\}\\
T_s^{\circ} &= {}^1/_2(p+q)h-{}^1/_2(p-q)h\cos 2\gamma-\tau h\sin 2\gamma-{}^1/_2\,\varepsilon\,[f'(\zeta)-\overline{f'(\zeta)}]\times\\
&\times[{}^1/_2(p-q)h(\zeta^2-\overline{\zeta^2})]-i\tau h(\zeta^2+\bar{\zeta}^2)]-{}^1/_2\,\varepsilon^2[f'(\zeta)-\overline{f'(\zeta)}]\times\\
&\times\{{}^1/_2(p-q)h(\zeta^2\overline{f'(\zeta)}-\bar{\zeta}^2f'(\zeta)-i[\zeta^2\overline{f'(\zeta)}+\bar{\zeta}^2f'(\zeta)]\tau h\}\\
S_{ns}^{\circ} &= -{}^1/_2(p-q)h\sin 2\gamma+\tau h\cos 2\gamma+{}^1/_2\,\varepsilon\,[f'(\zeta)-\overline{f'(\zeta)}]\,[{}^1/_2 i(p-q)\times\\
&\times h(\zeta^2+\bar{\zeta}^2)+\tau h(\zeta^2-\bar{\zeta}^2)]-{}^1/_2\varepsilon^2[f'(\zeta)-\overline{f'(\zeta)}]\times\\
&\times\{{}^1/_2(q-p)h(\zeta^2\overline{f'(\zeta)}+\bar{\zeta}^2f'(\zeta))+i\tau h[\zeta^2\overline{f'(\zeta)}-\bar{\zeta}^2f'(\zeta)]\}
\end{aligned}\tag{4.4}$$

As an example we shall consider the stress concentration near a square hole with rounded corners when the shell is extended along the generatrix. In this case $f(\zeta)$ in (2.1) should be taken in the form $f(\zeta) = \zeta^{-3}$, $\epsilon = \pm 1/9$, and it should be substituted into the differential operators (2.11).

Let us assume that the hole is free. Then the boundary conditions will be /103

$$T_n|_\Gamma = -T^{\circ}_n,\qquad S_{ns}|_\Gamma = -S_{ns}^{\circ},\qquad G_n|_\Gamma = 0,\qquad Q_n^{\circ}|_\Gamma = 0 \tag{4.5}$$

where T_n° and S_{ns}° are determined from (4.4) with $q = \tau = 0$ and $f(\zeta) = \zeta^{-3}$.

The zero approximation coincides with the solution for a circular hole (ref. 3), according to which $T_s^{(0)}$ has the form[1]

[1]Here we take into account the inaccuracy (ref. 10) permitted in reference 3.

$$T_s^{(0)} = ph[1 - 2\cos 2\gamma - {}^1/_2\pi\beta^2 \cos 2\gamma] \tag{4.6}$$

We shall include the coefficient in front of ϵ^j from T_s^o (4.4) in $T_s^{(j)}$.

From the boundary conditions (4.5) taking into account (4.4), (4.3), (2.11), we determine the constants contained in (4.3) when j = 1, 2

$$\begin{gathered}
A_2^{(1)} = -{}^1/_2 ph\pi r_0^2 n, \quad B_2^{(1)} = -A_2^{(1)}, \quad A_4^{(1)} = -A_2^{(1)}, \quad B_4^{(1)} = A_2^{(1)} \\
A_6^{(1)} = {}^1/_2 A_2^{(1)}, \quad B_6^{(1)} = -A_2^{(1)} \\
B_0^{(1)} = -{}^1/_8 ph\pi^2 r_0^2 n, \quad E_2^{(1)} = 3B_0^{(1)}, \quad F_2^{(1)} = -2B_0^{(1)}, \quad E_6^{(1)} = B_0^{(1)}, \\
F_6^{(1)} = -B_0^{(1)}, \quad A_0^{(2)} = -{}^3/_2 A_2^{(1)}, \\
A_2^{(2)} = A_2^{(1)}, \quad B_2^{(2)} = -2A_2^{(1)}, \quad A_6^{(2)} = {}^1/_2 A_2^{(1)}, \quad B_6^{(2)} = -{}^3/_2 A_2^{(1)} \\
A_8^{(2)} = -A_2^{(2)}, \quad B_8^{(2)} = {}^7/_2 A_2^{(1)} \\
A_{10}^{(2)} = A_2^{(1)}, \quad B_{10}^{(2)} = -18B_0^{(1)}, \quad E_2^{(2)} = 2B_0^{(1)}, \quad F_2^{(2)} = -{}^7/_2 B_0^{(1)} \\
E_4^{(2)} = 2B_0^{(1)}, \quad F_4^{(2)} = -2B_0^{(1)}, \quad B_0^{(2)} = -2B_0^{(1)} \\
E_6^{(2)} = 3B_0^{(1)}, \quad F_6^{(2)} = -{}^7/_2 B_0^{(1)}, \quad E_{10}^{(2)} = 4B_0^{(1)}, \quad F_{10}^{(2)} = -{}^9/_2 B_0^{(1)}
\end{gathered} \tag{4.7}$$

When j = 1, 2 the other constants in (4.3) are equal to zero. Taking into account (2.11) and (4.7) we find $T_s^{(1)}$ and $T_s^{(2)}$ from (4.3)

$$\begin{aligned}
T_s^{(1)} &= ph[-2\cos 2\gamma + 6\cos 4\gamma - 6\cos 6\gamma - \\
&\qquad - \pi\beta^2\, {}^1/_2 (1 - 3\cos 2\gamma + 3\cos 6\gamma)] \\
T_s^{(2)} &= ph[-2\cos 2\gamma - 6\cos 6\gamma + 18\cos 8\gamma + 18\cos 10\gamma + \\
&\qquad + {}^1/_4\pi\beta^2 (4\cos 2\gamma - 12\cos 4\gamma - 18\cos 6\gamma - 18\cos 10\gamma)]
\end{aligned} \tag{4.8}$$

According to (2.13) the value of $T_s^*|\Gamma$ in the second approximation has the form

$$T^*_{s\,(2)}\,|_\Gamma = T_s^{(0)} + \varepsilon T_s^{(1)} + \varepsilon^2 T_s^{(2)} \tag{4.9}$$

If $\epsilon = 1/9$, then the diagonal of the square will coincide with the direction of the generatrix; when $\epsilon = -1/9$ the diagonal of the square is at an angle of $1/2\ \pi$ with respect to the generatrix.

We present the values $k = T_s^*/ph$ on Γ when $\gamma = 1/2\ \pi$ for a plate and a shell obtained in the zero, first and second approximations, when

$$\gamma = {}^1/_2\pi, \qquad r_0/\sqrt{Rh} = 0.5, \qquad \varepsilon = {}^1/_9$$

Approximation	Zero	First	Second	Precise	
$k(\gamma = 1/2\ \pi) =$	+3.00	+4.55	+5.09	+5.38	for a plate
$k(\gamma = 1/2\ \pi) =$	+3.16	+4.80	+5.37	-	for a shell

The second approximation for the plate differs from the solution presented in reference 6 by 6 percent. In the case of the shell we may assume for practical purposes that the convergence rate is satisfactory, since the first approximation is 52 percent greater than the zero approximation, while the second approximation is 14 percent greater than the first approximation. /104

We present the values of k when $\gamma = 0$ and $\gamma = 1/2\ \pi$ for different values of $r_0/\sqrt{Rh}$ with $\nu = 0.3$.

	Plate	Shell		
$r_0/\sqrt{kh} =$	0.0	0.2	0.4	0.6
$k\|_{(\gamma=0)} =$	−1.31 −0.87	−1.34 −0.88	−1.53 −1.93	−1.81 −1.00
$k\|(\gamma = {}^1/_2\pi) =$	+5.09 +1.99	+5.12 +2.00	+5.28 +2.06	+5.52 +2.14

Here the upper values are for $\epsilon = 1/9$, while the lower values are for $\epsilon = -1/9$. We can see that the stress concentration coefficient depends on the curvature of the shell and that this relationship is particularly pronounced when $\gamma = 0$. Thus, within the considered limits of $r_0/\sqrt{Rh}$, the maximum increase in the stress concentration coefficient is greater than that for a case of a plate by a factor of 1.37 (i.e., when $r_0/\sqrt{Rh} = 0$). On the other hand, the increase in the maximum stress concentration coefficient is small (approximately 8 percent).

Figures 2 and 3 show the distribution of stresses $T_s{}^*$ on the contour of the hole when $\nu = 0.3$ and $r_0/\sqrt{RH} = 0.6$ for a shell and a plate. Let us consider briefly the case when the cylindrical shell is under tension and is weakened by a nonreinforced elliptical hole.

This problem is considered in detail in reference 11. Here we present only the value of $T_s{}^*|\Gamma$ for $\nu = 0.3$, taking into account the second approximation

$$T_s^*|_\Gamma = ph\left[1 - 2\cos 2\gamma + 2\frac{a-b}{a+b}(\cos 2\gamma - \cos 4\gamma) + 2\left(\frac{a-b}{a+b}\right)^2 \times\right.$$
$$\left.\times (\cos 4\gamma - 0.5\cos 2\gamma - 0.5\cos 6\gamma)\right] - 0.65\frac{(a+b)^2}{4Rh} ph\left[\cos 2\gamma + \right. \tag{4.10}$$

$$+\frac{a-b}{a+b}(1+\cos 4\gamma)+\left(\frac{a-b}{a+b}\right)^2(3\cos 2\gamma+\cos 6\gamma)\Big]$$

It follows from (4.10) that the qualitative difference in the distribution of forces in the shell compared with the plate will take place when $\gamma = 0$. Thus in this case for a plate we have $T_s^*|\Gamma = -ph$, independent of a/b, while in the shell /105

$$T_s^*|_{\gamma=0} = -ph\left\{1+0.65\frac{(a+b)^2}{4Rh}\left[1+2\frac{a-b}{a+b}+4\left(\frac{a-b}{a+b}\right)^2\right]\right\} \cdot \qquad (4.11)$$

This dependence will be substantial.

To estimate the rate of convergence of the obtained solution (4.10), we present the values of $k = T_s^*/ph$ for $\gamma = 1/2\,\pi$, $a/b = 1.4$, $1/2(a+b)/\sqrt{Rh} = 0.5$, $\nu = 0.3$ in the zero, first and second approximation.

Figure 4 shows the distribution of forces $T_s^*|\Gamma$ for a shell and a plate when the values of the parameters are as specified above. The maximum increase in the concentration coefficient in this case is attained when $\nu = 0$ and is equal to 23 percent, compared with its value for the plate. On the other hand, the increase in the maximum concentration coefficient is insignificant (5 percent).

Approximation	Zero	First	Second	Precise	
$k(\gamma = 1/2\,\pi) =$	+3.00	+2.32	+2.44	2.43	for a plate
$k(\gamma = 1/2\,\pi) =$	+3.16	+2.44	+2.57	-	for a shell

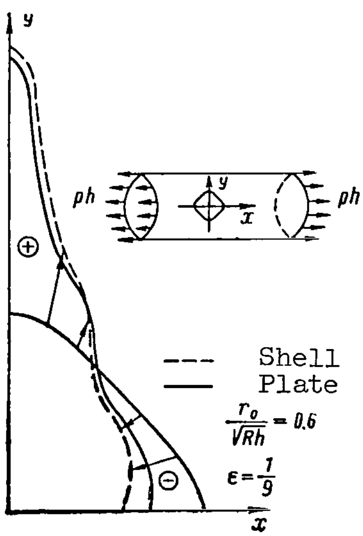

Figure 2

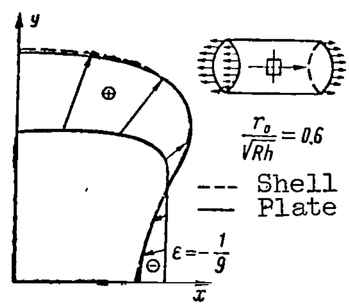

Figure 3

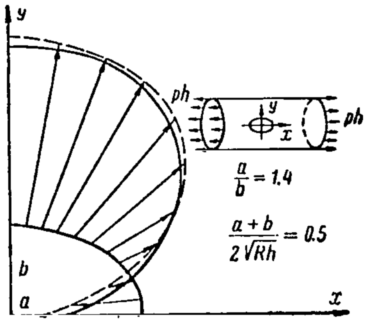

Figure 4

From the numerical examples which we have presented it follows that in the case of tension (compression) of a cylindrical shell along the generatrix, when the latter is weakened by nonsupported holes of arbitrary form, the maximum stress concentration coefficient in the shell differs little from the maximum stress concentration coefficient in a plate, when the loading is the same. To determine the maximum stress concentration coefficient during the tension of a cylindrical shell, weakened by a small unsupported hole of arbitrary shape, we can use the values of the stress concentration coefficients for the corresponding holes in a plate with an accuracy of 5 to 8 percent.

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